Statistical RKR Modeling of Mixed-Mode Fracture in a Brittle Functionally Graded Material

by

T. L. Becker, R. M. Cannon and R. O. Ritchie
University of California at Berkeley
and Lawrence Berkeley National Laboratory

USNCCM ‘99, Boulder, Co.
August 4, 1999
Outline

• LEFM solution and stress intensity factors for FGM’s
• Statistical Ritchie-Knott-Rice (RKR) modeling
• Finite element analysis and $K$-calibration for fracture mechanics sample with modulus gradient
• Calculate effect of gradient slope on
  • predicted fracture toughness, $K_\Phi$
  • average kinking direction, $\alpha$
Singular Crack Tip Fields in an FGM

- The singular stress field retains the strength and form of a homogeneous material [Erdogan, 1994]
- As \( r \to 0 \) in an FGM with
  \[ E(x) = E_o \exp(\beta x), \quad \nu(x) = \nu_o \]
  the stress field varies
  \[ \sigma_{ij} = \exp(\beta x) \left[ \frac{K_I f^I_{ij}(\theta)}{\sqrt{2\pi r}} + \frac{K_{II} f^II_{ij}(\theta)}{\sqrt{2\pi r}} \right] \]
  where \( K_I = \) mode-I S.I.F. (tensile mode)
  \( K_{II} = \) mode-II S.I.F. (shear mode)
- Similar to interface cracks, the \( K \) solutions for FGM’s depend on the material
The RKR fracture model correlates the onset of fracture with the development of a critical stress at a distance ahead of the crack tip.

A basis for this behavior is the influence of sampling volume on the measured strength of brittle materials. Using two-parameter Weibull statistics

\[
\Phi = 1 - \exp\left[ - \int_{\text{vol}} \left( \frac{\sigma}{\sigma_o} \right)^m \frac{dV}{V_o} \right]
\]

\(\Phi\) : total failure probability of a part
\(m\) : Weibull modulus
\(\sigma_o\) : scaling Weibull stress

Substituting singular crack tip stress field \((b \leq B, \text{total sample thickness})\)

\[
\Phi = 1 - \exp\left[ - \int_0^\pi \int_0^r \left( \frac{K}{\sigma_o \sqrt{2\pi r}} f_{ij}(\theta) \right)^m \frac{brdrd\theta}{V_o} \right]
\]

Lin, Evans and Ritchie (1986) use this methodology to describe the fracture behavior of low-toughness steels as a function of temperature.
Principal Question

Given:
1) Statistical RKR ⇒ Fracture of a brittle material can be calculated as function of stresses away from crack tip

2) FGM Crack-tip solution ⇒ Stresses are a function of modulus variation

What are the effects of modulus and strength gradients on the toughness and average kink direction of a brittle FGM?
Crack Tip Modeling

• $\Phi$ integral is not defined for $m > 4$ due to the strong singularity in $\sigma^m$!

• Lin, et al 1986 integrated the linear elastic stresses outside the plastic zone and nonlinear elastic stresses with a simplified blunting region.

• Here crack is modeled as a slender notch. Integration was performed over the sample, excluding a small near-notch zone with radius $\rho \sim a/10^5$

  • Results were weakly sensitive to the size of this zone

• For a given failure probability, toughness $K_\Phi$ can be calculated

\[
K_\Phi = \left[ \frac{-\ln(1 - \Phi)}{b \int (\bar{\sigma}_1 / \sigma_o)^m \, dA} \right]^{1/m}
\]

  (with $\bar{\sigma}_1 = \sigma_1 / K$)
Mean Location and Direction of Fracture

- For the Williams’ and HRR crack-tip fields, the most probably distance of fracture initiated, \( r^* \) has been calculated for the mode-I case.

- Similarly, for mixed-mode loading, determine the average location \( \{x,y\}, \{r,\alpha\} \) via a weighted Weibull integral:

\[
\begin{aligned}
\left\{ \bar{x}, \bar{y} \right\} &= \frac{\int \left\{ \frac{x}{\sigma} \right\}^m dA}{\int \left( \frac{\sigma}{\sigma_o} \right)^m dA} \\
\bar{r} &= \sqrt{\bar{x}^2 + \bar{y}^2} \\
\bar{\alpha} &= \text{ArcTan}(\bar{x}, \bar{y})
\end{aligned}
\]
Procedures

- Two gradient shapes were studied, allowing for a twenty-fold change in properties:
  - \( E(x), \sigma_0(x) = b \times +a \)
    - \( b = [-18, 18] \); \( a = 10.5 \)
  - \( E(y), \sigma_0(y) = (a-1) \tanh(b \times y) + a \)
    - \( b = [0, 5] \); \( a = 10.5 \)
- Plane strain; Poisson’s ratio, \( \nu = 0.3 \).
- Calculations were performed for a SEC(T) sample with a single crack length, \( a/W = 0.5 \), \( W = 1 \).
- \( K \)-calibrations were performed for each gradient considered.
Numerical Procedures

• Finite element code FEAP 4.2 (Zienkiewicz & Taylor, 1987) used with plane-strain linear elastic finite element such that elastic constants were varied quadratically within a single element.

  • Element formulation checked against solution of rigid indentation of FGM (Kassir, 1974).
  • Crack tip modeled with 40 singular Stern & Becker triangular elements in fan array.
  • 2300 total elements, 9333 nodes.

• Weibull integral was calculated from FEA viz.

\[ \sum_{j=1}^{\text{elems}} \sum_{i=\text{Gauss pts}}^{m} \int_{0}^{b} \frac{dV}{V_{o}} \approx \frac{b}{\sigma_{o} V_{o}} \left[ \sum_{i}^{m} \int_{v_{o}}^{V_{o}} \sum_{j}^{\text{h}} \epsilon_{i}^{m} w_{i} \right] \]
Stress Intensity Factors for $E(x)$ and $E(y)$

- For a homogeneous SEC(T), $K = f(P, a/W)$, independent of modulus. For an FGM $K = f(P, a/W, bW)$ needs to be determined.
- $K_I, K_{II}$ for each gradient obtained by fitting the stresses ahead of the crack.
- The $E(x)$ results indicate that the crack tip is shielded when entering stiffer material.
- For comparing the different gradients, failure probabilities can be calculated for the same applied load $P$, or for the same applied $K$.
- Changing the basis for comparison will reverse the trends observed.
- For $E(y)$ increase in gradient slope increase phase angle
Predicted Fracture Toughness

- Linear gradients in $x$
- Mode-I loading in all cases
Predicted Kink Angle

- Gradient in Weibull Strength, $\sigma_o(y)$
- Far-field & near-tip mode-I loading only, $K_{II}=0$
Predicted Kink Angle

- Gradient in modulus, $E(y)$
- Far-field mode-I loading; near-tip mixed mode, $K_I$ & $K_{II}$
Stress Field with Modulus Gradient
Summary

- Finite element calculations indicate stress intensity shielding for cracks in an FMG with a positive modulus slope.
- Current model predicts expected fracture toughness will increase for cracks growing into a more compliant material.
- Kinking analysis predicts sharp kinks in FGMs with strength gradients and Weibull moduli, $m<7$.
- For FGMs with $E(y)$, nominal mode-I loading results in mixed-mode loading at the crack tip.
- For very low Weibull modulus materials, kinking analysis predicts trends opposite to that dictated by near-tip considerations.