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# On the theoretical modeling of fatigue crack growth

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## ABSTRACT

Although fatigue is by far the most common mode of failure of structural materials, mechanistic understanding is still lacking. For example, the fundamental Paris law which relates the crack growth rate to stress-intensity factor range is still phenomenological and no reliable mechanistic model has been established for a given material or class of materials despite numerous investigations over a half a century. This work is an attempt to theoretically model fatigue crack propagation induced by alternating crack-tip plastic blunting and re-sharpening in the mid-range of growth rates on the basis of inputs from experiments that measure macroscopic material behavior, e.g., response to uniaxial cycling loading. In particular, we attempt to predict Paris law behavior by accounting for the material constitutive behavior in response to cyclic loading by modeling crack advance solely in terms of the underlying plastic dissipation. We obtain the steady-state condition for crack growth based on plastic dissipation, numerically using finite element analysis, which involves a methodology to address plastic closure upon unloading. For a given stress-intensity range, we calculate the crack propagation rate from the steady-state condition through each cycle of loading and unloading of a cracked compact-tension specimen, without resorting to any specific criterion for crack advance.

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#### 1. Introduction

Fatigue is the most ubiquitous mode of fracture in metallic materials accounting for more than 80% of all *in service* failures in structural materials, yet paradoxically it is the least understood form of fracture mechanistically and the least modeled from a mechanics perspective. The process of fatigue failure is one of crack initiation and subsequent crack growth driven by the presence of cyclically varying loads. Accordingly, life prediction strategies are based on either total life, including both the crack initiation and propagation stages, in terms of the stress/strain-life (S-N) approach, or one based solely on the crack propagation life which assumes the inevitable presence of pre-existing defects and estimates the remaining life in terms of the number of cycles for the largest undetected crack to grow to failure, *e.g.*, at the limit load or when the maximum stress intensity,  $K_{max}$ , exceeds the fracture toughness,  $K_c$ . The latter life-prediction approach is termed the

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https://doi.org/10.1016/j.jmps.2018.07.026 0022-5096/Published by Elsevier Ltd. damage-tolerant approach and relies on the use of fracture mechanics; in principle, it is more conservative than the S-N approach and, as such, is used to predict the remaining safe lifetime for most safety-critical applications.

To utilize the damage-tolerant approach, a knowledge of the rate of propagation of fatigue cracks is the main key parameter, along with the applied stresses, crack sizes and shape, and the geometry of the component. The most notable attempt to relate the crack-growth rate per cycle (da/dN) to the applied stresses was proposed by Paris and coworkers (Paris et al., 1961; Paris and Erdogan, 1963) in the form of the so-called Paris power law, where da/dN was related to the stress-intensity range  $\Delta K$ , under nominally small-scale yielding (SSY) conditions, by:<sup>1</sup>

$$\frac{da}{dN} = C \ \Delta K^m,\tag{1}$$

where  $\Delta K = K_{\text{max}} - K_{\text{min}}$ , the difference between the maximum and minimum stress intensities in a given loading cycle, and *C* and *m* are scaling constants. For the mid-range of growth rates, typically from ~10<sup>-6</sup> to 10<sup>-3</sup> mm/cycle, the Paris exponent *m* typically takes values between ~2 and 4 in metallic materials; *m* values can be significantly higher in intermetallics and ceramics (Ritchie, 1999). Although the Paris law is purely empirical and provides no insight into the mechanism of fatigue-crack propagation, it can successfully utilize experimental growth rate results from "coupon" tests in the laboratory to predict the fatigue-crack growth rates in structures and components in service, strictly for large cracks<sup>2</sup> subjected to constant-amplitude cyclic loading under SSY conditions.

Since the introduction of Paris law, many studies on fatigue crack growth have attempted to experimentally relate crack growth rates to the stress intensity *K*, or more precisely to the stress-intensity range,  $\Delta K$ . Far fewer studies, however, have attempted to model fatigue crack growth analytically based on a failure criterion for crack advance and as such correlate the crack growth rate to the stress-intensity factor. Of note in this regard are the works of Rice (1967) and Weertman (1969, 1973), in which a critical energy criterion for crack advance was employed; these models predict a Paris exponent of m = -4. Alternative modeling based on a cumulative damage criterion (Liu and Lino, 1969; Majumdar and Morrow, 1974; Saxena and Antolovich, 1975), *e.g.*, the Coffin–Manson relationship (Coffin, 1954; Manson, 1953), has also been used to theoretically predict the *da/dN* vs.  $\Delta K$  relationship; these models also tend to predict m = 4. There have also been numerous other approaches based strain-controlled crack advance (Purushothaman and Tien, 1978; Tien and Purushothaman, 1978), cohesive zone treatments (Tomkins, 1968), and irreversible damage accumulation (McCartney and Gale, 1971).

A different, more mechanistic, approach was followed by McClintock (1967) who demonstrated through dimensional analysis that the crack growth rate could be related to the cyclic crack-tip opening displacement ( $\Delta$ CTOD) based on a cracktip blunting mechanism (Laird and Smith, 1962). The crack-tip blunting mechanism had been theorized by Laird and Smith (1962), Pelloux (1970) and Neumann (1974) based on experimental observations; it states, we believe quite accurately, that mode  $I^3$  cyclic fatigue-crack advance in a ductile metal is the result of alternating crack-tip blunting during the loading cycle followed by re-sharpening of the crack tip during unloading. These models are consistent with the creation of fatigue striations, which are the characteristic fracture mode for fatigue crack growth in ductile materials. Subsequently, several attempts (Gu and Ritchie, 1999; Tvergaard and Hutchinson, 2002; Tvergaard, 2004) have been made to model fatigue-crack growth based on this  $da/dN \propto \Delta$ CTOD mechanism; in contrast to the damage-accumulation models which tend to predict a Paris exponent of m=4, as the CTOD scales with  $K^2$ , these approaches tend to predict a Paris exponent of m=2. Using the mechanistic hypothesis that crack advance by alternating crack-tip blunting and resharpening relies solely on cyclic plasticity, Gu and Ritchie (1999) used finite element analysis to model such plasticity-driven crack propagation of a blunted crack under cyclic loading based on crack-tip blunting without the need to introduce any specific failure criterion. Their numerical studies were carried out for only two load cycles to avoid very strong mesh distortion around the crack tip due to accumulation of large plastic deformation. According to their results, the crack growth rate was found to be proportional to the  $\triangle$ CTOD, as  $da/dN = \beta \triangle$ CTOD with the proportionality constant of  $\beta = 0.33$  for  $da/dN > 10^{-4}$  mm/cycle. Although the numerical fatigue crack growth model was idealized by considering only two load cycles with no considerations of crack closure, it reproduced the crack-tip blunting mechanism which, as noted above, is considered as the fundamental mechanism for plasticity-driven crack growth under cyclic loading. Tvergaard (2004) extended the work of Gu and Ritchie by incorporating a remeshing technique, and accordingly was able to similarly model such plasticity-driven crack growth but now through many load cycles, specifically for an initially sharp crack in a center-cracked tension specimen of a perfect plasticity material. However, Tvergaard (2004) only considered the results of the first few cycles in investigating the rela-

<sup>&</sup>lt;sup>1</sup> This is the Paris law in its simplest form; it describes principally the mid-range of growth rates in the range  $\sim 10^{-6}$  to  $10^{-3}$  mm/cycle. However, additional terms can be added to allow for such parameters as mean stress, frequency and environment, or to account for the deceleration of growth rates as the fatigue threshold,  $\Delta K_{\text{TH}}$ , is approached, or the acceleration at high growth rates when  $K_{\text{max}}$  approaches  $K_c$  (Ritchie, 1977).

<sup>&</sup>lt;sup>2</sup> The term "large" here means large compared to the scale of microstructure, the extent of local crack-tip plasticity (*i.e.*, the plastic-zone size), and/or the scale of crack-tip shielding (*e.g.*, crack closure) in the wake of the crack. In contrast, so-called "small" cracks, which do not meet one or more of these size criteria, can display "anomalous" behavior (Suresh and Ritchie, 1984; Ritchie and Lankford, 1986).

<sup>&</sup>lt;sup>3</sup> The arguments described here pertain to fatigue-crack growth in mode I, *i.e.*, along a path nominally perpendicular to the maximum tensile stresses (where  $K_{II} = 0$ ). The vast majority of experimental and theoretical studies on fatigue-crack growth have been carried out for this situation as this is invariably how fatigue cracks propagate in service. However, although much rarer, fatigue cracks can grow in mode II (shear) and mode III (anti-plane shear), *e.g.*, under torsion loading in the presence of ample plasticity, and a few theoretical models, similarly based on alternating crack-tip displacements or damage-accumulation, have been proposed for such non-mode I crack advance. If interested, the reader is referred to ref. Nayeb-Hashemi et al. (1983).

Several different approaches have been taken to model fatigue crack propagation numerically. They can be broadly categorized into approaches that use: (i) a nodal release technique, (ii) cohesive zone modeling, and (iii) utilize a damage accumulation criterion. In the first category, fatigue crack growth is modeled through finite element analysis by releasing the crack-tip node such that the crack advances by the crack-tip element size each time the node is released. The nodal release technique for fatigue crack growth modeling was originally proposed by Newman (1976) to investigate crack closure behavior in a center-cracked panel under cyclic loading. Newman utilized a nodal release at the maximum applied load in each fatigue cycle. Although mechanistically arbitrary, many researchers, e.g., (McClung and Sehitoglu, 1989; Ellyin and Wu, 1992; McClung and Sehitoglu, 1992; Wu and Ellvin, 1996; Roychowdhury and Dodds, 2003; Solanki et al., 2004; Kiyak et al., 2008). have used a similar approach and released nodes at different specific loading points, e.g., at the maximum or minimum applied load, or somewhere in between, to study the effect of the crack advance per cycle. However, as the crack growth increment per cycle is dictated by the element size, fatigue crack growth rates cannot be predicted by this approach. Nevertheless, the method has been used to study several aspects of crack closure in fatigue and to obtain a description of the plastic wake zone and crack-tip opening displacements (Kiyak et al., 2008) by releasing the nodes such that the crack advance is equated to an experimentally measured value. There have been also some attempts to advance the crack by the nodal release technique based on a specific criterion; for example, Newman (1977) incorporated a critical strain criterion whereas Nakagaki and Atluri (1980) used a critical stress criterion, in both cases to extend the crack by nodal release.

In the second category, fatigue crack growth is simulated through use of a cohesive zone model in which the fracture process is assumed to occur in a thin region ahead of the crack tip; crack growth is considered as a consequence of continuous debonding of the cohesive zone material. To investigate the debonding behavior, a constitutive law for the propagation of the cohesive zone tip is required. Since the original cohesive zone model (Dugdale, 1960) was proposed for monotonic loading, it has been necessary to develop specific cohesive zone models suitable for cyclic loading in order to apply them to the phenomenon of fatigue crack growth. Despite these drawbacks, several studies have used a cohesive zone approach to model fatigue crack growth (De-Andrés et al., 1999; Nguyen et al., 2001; Roe and Siegmund, 2003; Maiti and Geubelle, 2006; Bouvard et al., 2009; Ural et al., 2009; Harper and Hallett, 2010). De-Andrés et al. (1999) modeled fatigue crack growth through a linear unloading-reloading cohesive law without cyclic degradation of the stiffness. Nguyen et al. (2001) argued that such a model can result in plastic shakedown after a few cycles and therefore proposed a cohesive law for cyclic loading by considering the distinction between the loading and unloading paths. They investigated the ability of their model to replicate experimental results of fatigue crack growth rate measured in an aluminum center-cracked sheet sample. Roe and Siegmund (2003) introduced a damage variable in their proposed cohesive law to consider the degradation of the cohesive zone traction during cyclic loading. They investigated fatigue crack growth of interface cracks by considering the crack advance to result from the competition between material debonding and plastic deformation processes. Following Roe and Siegmund (2003), several researchers, e.g., (Maiti and Geubelle, 2006; Bouvard et al., 2009; Ural et al., 2009; Harper and Hallett, 2010), proposed a cohesive law described by damage evolution specifically to model fatigue crack growth.

Although a few attempts have been undertaken to model fatigue crack growth using the cohesive zone approach in metals such as aluminum (De-Andrés et al., 1999; Nguyen et al., 2001; Ural et al., 2009), the approach is widely used in modeling fatigue along interfaces (Roe and Siegmund, 2003), in polymer materials (Maiti and Geubelle, 2006), in single crystal superalloys (Bouvard et al., 2009), and in composite materials (Harper and Hallett, 2010). The idea is that the underlying principle in cohesive zone models is decohesion which is stress-controlled. However, fatigue crack growth in ductile materials is in essence plasticity-driven and so decohesion-based models are inconsistent with the fundamental mechanisms responsible for cyclic crack advance. In light of this, we believe that any physical justification for the use of cohesive zone modeling of fatigue crack growth in ductile materials is deficient. Moreover, additional calibrations are required to determine the parameters for cohesive zone models specific to cyclic loading.

As noted above, damage accumulation criteria have also been used to model fatigue crack growth. In this approach, incremental damage accumulation leads to fatigue crack advance when a critical amount of damage is reached ahead of the crack tip. Several researchers have proposed damage criteria based on fatigue ductility (Oh, 1978), plastic strain (Glinka, 1982; Radon, 1982) or a measure of strain energy density at the crack tip (Majumdar and Morrow, 1974; Kujawski and Ellyin, 1984, 1987; Jiang, 2000). As an example, Kujawski and Ellyin (1987) proposed a damage accumulation criterion based on the product of the plastic strain range and stress range. They adopted Rice's analytical solution (Rice, 1967) to determine stress and strain distributions ahead of the crack tip under cyclic loading and managed to get a good agreement with the experimental fatigue crack growth data (Stephens et al., 1984) by adjusting the damage process-zone size which was a material parameter in their model.

One of the more promising models for the prediction of fatigue crack growth based on damage accumulation was proposed by Jiang and Feng (2004). They adopted the damage criterion of Jiang (2000), in which the damage parameter is considered to be proportional to the plastic work. To calculate the damage parameter, they used stress and strain distributions ahead of a stationary crack under a given cyclic loading obtained from finite element analysis with the use of the plasticity model suggested by Jiang and Sehitoglu (1996). The model of Jian and Sehitoglu is a modified Armstrong-Frederick plasticity model that better addresses the ratchetting response under multiaxial loading conditions. They assumed that the crack advances once the fatigue damage accumulation reaches a certain critical value and developed a relationship for the crack growth rate in terms of the damage per cycle ahead of the crack tip. They designated the final crack growth rate as the rate corresponding to the 10<sup>th</sup> cycle, as they found the calculated damage per cycle ahead of the crack tip stabilized after 10 cycles.

Since many of the models described above depend upon arbitrary assumptions of when crack advance occurs, or utilize failure criteria which are not necessarily supported by mechanistic and experimental studies, we present an approach here for calculating fatigue crack growth rates in ductile materials under mode I cyclic loading which, akin to the earlier approaches of Gu and Ritchie (1999) and Tvergaard (2004), is based solely on cyclic plasticity and requires no specific criteria for failure. In our approach, as elaborated in Section 2, we first establish the stress and deformation fields under cyclic loading in the vicinity of a fatigue crack and then compute the rate of growth of the crack based on the mechanism of crack-tip blunting on the loading cycle followed by resharpening on unloading, *i.e.*, by ductile striation crack growth. The method is demonstrated for a compact-tension specimen under cyclic loading in Section 3. It is noted that fatigue crack growth can be affected by a multiple of additional mechanisms, the most important of which are the occurrence of void growth or microcracking ("static modes") ahead of the crack tip at high crack growth rates and crack wedging resulting from crack closure mechanisms behind the crack tip, in particular at low, near-threshold, crack growth rates (Ritchie, 1988). However, to elude complexity from considering these additional mechanisms, in the present work we only consider the crack propagation due to the plastic deformation including the effects of the associated (plasticity-induced) crack closure upon unloading.

It is noted that although the modeling of fatigue crack growth has received much attention over the years, a sound description of stress and strain fields ahead of a propagating fatigue crack by considering an appropriate constitutive law for cyclic loading has been lacking. A few attempts have been made to describe the stress and strain near a fatigue crack tip, but they are based on the fields near a stationary crack (Ellyin and Wu, 1992; Wu and Ellyin, 1996), or they utilize  $J_2$ -plasticity theory to analyze large plastic deformation near a fatigue crack, *e.g.*, (Roychowdhury and Dodds, 2003; Toribio and Kharin, 2009), which may not be an appropriate material model for cyclic loading. A unique aspect of the present approach is that we analyze the stress and strain fields near a propagating crack by incorporating a proper constitutive model for cycling loading.

# 2. Model formulation

In this section we present our approach to determining the steady-state condition which we will use to calculate the crack propagation rate for a given stress-intensity range. Along with our depiction of the process that we use to obtain the steady-state condition, we also present the constitutive model that describes the cyclic plasticity of our model material.

#### 2.1. Methodology for the calculation of crack propagation rate

To discern the plasticity-driven crack propagation rate under cyclic loading, our method involves two principal steps: (i) the steady-state condition for a propagating crack is found for a given magnitude of cyclic loading, and (ii) with the material at the crack tip experiencing the steady-state stress and strain fields, the crack advance due to one cycle of load is calculated. This crack advance is designated as the fatigue crack growth rate under the applied cyclic load condition. The steady-state condition around a fatigue crack is obtained through finite element analysis using the nodal release technique. Once the steady-state condition is reached, the nodal release is stopped, one further cycle of load is applied to the crack and the crack growth rate is calculated, as mentioned above. Here we would like to emphasize that continued cycling loading beyond the steady-state condition (without nodal release) would lead to continuous and unrealistic plastic strain accumulation, as elaborated in Appendix A for the stationary crack. Cognizant of this fact, we determine the crack growth rate through only one additional cycle imposed on the steady state (with no nodal release). By doing so, we believe that we model the blunting and re-sharpening mechanism of growth in the most realistic way within the framework of our model, whereby the material ahead of the crack tip experiencing the additional cycle is not sent back to the crack flanks as is the case in real world situations. In other words, we consider that the additional cycle on the top of the steady-state condition simulates, in an approximate and physical way, the steady-state process of blunting and re-sharpening of the crack tip, namely the very mechanism of crack advance. Below, we describe our specific approach to obtaining the steady-state condition.

## 2.2. The steady-state condition for a propagating crack under cyclic loading

For a pre-crack in a test specimen subjected to cyclic loading, the crack can start growing and eventually reach a constant crack growth rate under a given cyclic load. This constant crack growth rate represents the "steady-state condition" whereby the stress and deformation fields around the crack tip under a given state of loading, *e.g.*, at peak loading, do not change as the crack advances. Note that while these crack-tip fields change as the load increases or decreases, when the applied load reaches the same state in terms of value and direction of loading, the crack-tip fields will be the same as the crack advances. In the present work, we use the nodal release technique to advance the crack with the use of an appropriate nodal release criterion until the steady-state condition is reached. The nodal release criterion is based on the accumulated plastic work as we will elaborate in the next subsection. We would like to draw attention to the fact that the steady-state condition for the stress and strain fields that we obtain numerically should be independent of the element size and the nodal release



Fig. 1. Schematic of the numerical crack-advance method by nodal release when the accumulated plastic work density at the crack-tip element reaches a critical value.

criterion. We would also like to note that continued application of cyclic loading to a stationary crack leads to continued plastic straining (Appendix A).

#### 2.3. Criterion for nodal release

The notion of crack extension on the basis of accumulated plastic work has been used by several researchers to model fatigue crack growth (Rice, 1967; Weertman, 1969, 1973; Jiang and Feng, 2004). Here we consider a similar criterion and advance the crack numerically by nodal release once the accumulated plastic work density at the crack-tip element reaches a critical value. The accumulated plastic work density at the crack-tip element  $\bar{W}_{\text{cracktip}}^p$  can be calculated as

$$\bar{W}_{\text{cracktip}}^{p} = \frac{1}{A} \int_{A} \int_{t} \sigma_{ij} D_{ij}^{p} dt dA, \qquad (2)$$

where  $\sigma_{ij}$  and  $D_{ij}^p$  are respectively the components of the Cauchy stress and plastic deformation rate tensors, *A* is the cracktip element area, *t* is the time, and a repeated index implies standard summation over the range. It should be noted that the accumulated plastic work density at the crack-tip element is just a representation of the plastic work that is dissipated at the crack tip. As we will elaborate below in Section 4, one can obtain the same steady-state condition even if all the elements in the plastic zone, or a row of elements ahead of the crack tip, are used to represent the crack-tip plastic dissipation in the criterion for the nodal release.

When the accumulated plastic work density at the crack-tip element reaches a critical value  $D_0$ , the crack-tip node is released. Fig. 1 shows a schematic of the nodal release method in which the crack-tip node, point A, is released when the criterion is met. During the nodal release process, the applied external load is kept constant while the nodal force on the crack-tip node is brought to zero. Once the crack-tip node is released, the crack advances by one element and a new crack-tip node is established (point B). The process is continued with change in load and concurrent monitoring of the accumulated plastic work at the new crack-tip element until the criterion for nodal release is met again.

#### 2.4. Constitutive law for cyclic plasticity

Material ahead of a fatigue crack experiences, respectively, both tension and compression deformation during the loading and unloading cycle. Therefore, an appropriate model for the simulation of fatigue crack growth must exhibit such material response features under cyclic loading. Depending on how the dislocation substructures differ under monotonic vs. cyclic loading, most metals subjected to cyclic loading will undergo either cyclic hardening or cyclic softening with subsequent material stabilization after a certain number of cycles (Morrow, 1965; Tuegel, 1987). This can be interpreted as stabilized yield surface after a certain number of cycles. Another cyclic loading feature is ratcheting observed under cyclic loading with nonzero mean stress. An appropriate model for cyclic loading should thus have the capability to predict both these cyclic hardening/softening and ratchetting phenomena. The stabilization of the yield surface with cyclic loading can be modeled by adding an isotropic hardening component to the flow stress. Further, as it has been shown that material models formulated with kinematic hardening rules can exhibit ratcheting behavior (Bari and Hassan, 2000), the description of ratchetting here is introduced into the constitutive model through kinematic hardening. One of the most prominent nonlinear kinematic hardening models was developed by Armstrong and Frederick (1966). Subsequent to their work, several models were proposed based on their approach to improve the ratcheting prediction under different cyclic loading conditions, *e.g.* (Jiang and Sehitoglu, 1994; Chaboche and Nouailhas, 1989; Chaboche, 1991; Ohno and Wang, 1993). Notable among these revised approaches was that of Chaboche (1991); indeed, we use the Chaboche constitutive model here as it is arguably the most comprehensive model which is capable of capturing many prominent features induced by cyclic loading, such as plastic shakedown, ratcheting, and the relaxation of the mean stress. In the following discussion, we briefly present a general formulation of the Chaboche material model.

The deformation rate tensor **D** can be decomposed into the elastic  $\mathbf{D}^{p}$  and plastic  $\mathbf{D}^{p}$  deformation rate tensors as:

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p. \tag{3}$$

The elastic behavior is assumed to be isotropic and linear such that:

$$\stackrel{\nabla}{\boldsymbol{\sigma}} = \mathbf{C}^e : \mathbf{D}^e, \qquad \mathbf{C}^e = 2G \,\mathbf{K} + \frac{(1+\nu)}{(1-2\nu)} 2G \,\mathbf{J},\tag{4}$$

where  $\stackrel{\nabla}{\sigma}$  is the Jaumann derivative of the Cauchy stress tensor  $\sigma$ ,  $\mathbf{C}^e$  is the fourth-order elasticity tensor, G and  $\nu$  are respectively the shear modulus and the Poisson's ratio,  $\mathbf{J} = \delta \delta/3$ ,  $\delta$  is the second-order identity tensor whose components  $\delta_{ij}$  are the Kronecker delta,  $\mathbf{K} = \mathbf{I} - \mathbf{J}$ , and  $\mathbf{I}$  is the symmetric fourth-order identity tensor. Then the elastic component of the deformation rate tensor can be expressed as:

$$\mathbf{D}^{e} = \frac{1}{2G} \stackrel{\nabla}{\mathbf{\sigma}} - \frac{\nu}{1+\nu} \frac{1}{2G} \stackrel{\nabla}{\boldsymbol{\sigma}}_{kk} \boldsymbol{\delta}.$$
(5)

Isotropic yielding is described by:

$$f(\boldsymbol{\sigma}, \boldsymbol{\alpha}, R) = \Sigma_e - \sigma_0 - R = 0, \tag{6}$$

where  $\Sigma_e = \sqrt{(3/2)(\sigma' - \alpha) : (\sigma' - \alpha)}$  is the effective stress,  $\sigma'$  is the deviatoric stress tensor such that  $\sigma'_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$ ,  $\alpha$  is the back stress tensor,  $\sigma_0$  is yield stress, R describes either isotropic hardening if positive or isotropic softening if negative,  $\mathbf{A} : \mathbf{B} = A_{ij}B_{ij}$  where  $\mathbf{A}$  and  $\mathbf{B}$  are second order tensors, and the standard summation convention is implied over a repeated index unless is specified otherwise. The back-stress tensor  $\alpha$  is composed of three parts (*i.e.*,  $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ ); according to the model by Chaboche, the time evolution of this tensor is given by:

$$\stackrel{\nabla}{\boldsymbol{\alpha}_{i}} = \frac{2}{3} C_{i} \mathbf{D}^{p} - \gamma_{i} \boldsymbol{\alpha}_{i} \dot{\varepsilon}^{p}, \quad i = 1, 2, 3 \text{(no summation over } i\text{)}$$
(7)

where  $C_i$  and  $\gamma_i$  are material constants and

$$\dot{\varepsilon}^p = \sqrt{2\mathbf{D}^p : \mathbf{D}^p/3} \tag{8}$$

is the effective plastic strain rate. The associated flow rule is written as:

$$\mathbf{D}^p = \dot{\varepsilon}^p \mathbf{N},\tag{9}$$

in which N is the normal to the yield surface, i.e.,

$$\mathbf{N} = \frac{\partial f(\boldsymbol{\sigma}, \boldsymbol{\alpha}, R)}{\partial \boldsymbol{\sigma}} = \frac{3}{2} \frac{\boldsymbol{\sigma}' - \boldsymbol{\alpha}}{\Sigma_e}.$$
(10)

Lastly, the time evolution of the isotropic hardening variable *R* is stated as:

$$\dot{R} = h(Q - R)\dot{\varepsilon}^p,\tag{11}$$

where *b* and *Q* are material constants. The parameter *R* increases monotonically with plastic strain and levels to the value of *Q*. In summary, the Chaboche material model for cyclic plasticity that we adopted in this work involves nine parameters:  $C_1$ ,  $C_2$ ,  $C_3$ ,  $\gamma_1$ ,  $\gamma_2$ ,  $\gamma_3$ , *Q*, *h*, and  $\sigma_0$ . Specific attention should be drawn to the parameter  $\gamma_3$ . If it is zero, the model can be used to capture plastic shakedown in a stress-controlled test, whereas if it is not zero the model can describe ratchetting at a constant plastic strain rate.

# 3. Numerical implementation

We present the application of our model for the prediction of fatigue crack growth rates in a cyclically-loaded compacttension C(T) specimen. In our calculations, the width and height of the specimen are W = 26.6 mm and 2H = 1.25 W, respectively; the C(T) specimen has an initially sharp crack with an initial length  $a_0 = 7.1$  mm. The geometry and all dimensions of



Fig. 2. Schematic of the compact-tension C(T) specimen loaded under cyclic loading.



**Fig. 3.** The finite element mesh for fatigue crack growth modeling in the C(T) specimen: (a) overall discretization of the domain, (b) near-tip mesh layout, (c) near crack-tip mesh detail. All coordinates are in mm.

the C(T) specimen are shown in Fig. 2. We carry the calculations under plane strain. Due to symmetry in loading and geometry, we analyze only the upper half of the specimen; on the symmetry line ahead of the crack tip, a symmetry boundary condition is imposed and the crack face is traction free if closure has not taken place. The specimen is loaded by incrementally changing a force applied as a distributed load along the load line (see Fig. 3a). The stress-intensity factor is calculated in terms of the applied force per unit thickness *F* through (Anderson, 2017):

$$K_I = \frac{F}{\sqrt{W}} f(\frac{a}{W}),\tag{12}$$

where *W* is the width of the specimen, *a* represents the crack length measured from the load line with  $a = a_0 = 7.1$  mm in the undeformed state, and *f* is the geometry function for the C(T) sample, which is a function of the crack length to width ratio, *a*/*W*, given by:

$$f(\frac{a}{W}) = \frac{(2 + \frac{a}{W})}{(1 - \frac{a}{W})^{3/2}} \left[ 0.886 + 4.64(\frac{a}{W}) - 13.32(\frac{a}{W})^2 + 14.72(\frac{a}{W})^3 - 5.6(\frac{a}{W})^4 \right].$$
(13)

In the simulations, we considered the possibility of crack closure<sup>4</sup>. Crack closure (Elber, 1970, 1971) is an important phenomenon in the propagation of fatigue cracks as it results in contact between the crack surfaces in the wake of the crack tip, and as such acts to "shield" the crack from the applied driving force, *i.e.*, the local stress intensities actually experienced at the crack tip are smaller than those associated with the applied load (Ritchie, 1988). Here we consider crack closure generated solely from the presence of cyclic plasticity – this is termed plasticity-induced crack closure. The implementation of such closure in our simulations was performed as follows:

- 1 For a given crack length, the solution to the boundary value problem of cyclic elasto-plasticity was carried out by incrementally changing the applied force *F*: knowing the stress and strain fields at time  $t_n$ , we obtain the solution at time  $t_{n+1}$ . During unloading ( $\Delta F < 0$ ), we check the coordinates of all nodes on the crack face at time  $t_{n+1}$  for possible crack closure. Crack closure occurs if a node has crossed the symmetry line, and the node is therefore designated as a closure node. In that case, the procedure for the solution at time  $t_{n+1}$  is repeated with the corresponding zero traction condition replaced with a zero-displacement condition at the closure node.
- 2 When the loading direction is reversed from unloading to loading, if there has been crack surface contact, *i.e.*, closure during the previous unloading, we monitor the force on the closure nodes. Since the contact nodes are under compression, change of the nodal force at any contact/closure node from compression to tension during the incremental solution process indicates the end of contact for that node. We repeat the solution process for the last time step (end of contact) with the zero-displacement boundary condition changed to zero-traction boundary condition on the corresponding node of the freshly exposed part of the surface.

Fig. 3 shows the finite element mesh we used to discretize the C(T) specimen. To capture the severity of the plastic deformation near the crack tip, the mesh was refined with a uniform element size around the crack tip, as shown in Fig. 3b and c. We solved the finite-strain elastic-plastic boundary value problem using standard finite element formulation (Liang and Sofronis, 2003). We assumed homogeneous material with the Chaboche constitutive law presented in Section 2.4. The constitutive equations were integrated as described in detail in Appendix D. In order to minimize numerical error due to severe mesh distortion upon loading and unloading as the crack advances, the mesh layout ahead of the crack tip was continuously adjusted if severely strained. The re-meshing procedure was performed when the accumulated plastic strain from the last re-meshing reached a critical value, which we considered as 0.4. When the re-meshing condition is met, a new mesh was generated and the stress and strain fields as well as state variables at the new integration points of each element were calculated from projection of those values from the distorted mesh (Tvergaard, 2004). The procedure for re-meshing is elaborated in Appendix B.

We carried out simulations with a model material whose constitutive response was described in Section 2.4 and its parameters are given in Table 1. Details of the model's response in uniaxial cycling straining are presented in Appendix C.

# 3.1. Critical plastic dissipation for nodal release

An estimate of the critical plastic dissipation parameter in the nodal release criterion can be obtained utilizing the well-known cumulative damage model defined by the Coffin–Manson relationship (Coffin, 1954; Manson, 1953). The Coffin–Manson model provides the fatigue life  $N_f$  of (invariably) an unnotched specimen under uniaxial tension/compression plastic strain loading as a function of the plastic strain amplitude  $\Delta \varepsilon^p/2$ :

$$\frac{\Delta\varepsilon^p}{2} = \varepsilon_f' (2N_f)^c,\tag{14}$$

<sup>&</sup>lt;sup>4</sup> Note that there are other, more potent, forms of crack closure that originate at low, near-threshold, stress intensities associated with wedging between crack surfaces; these mechanisms become active as the CTODs become comparable with the thickness of the entity that is wedging the crack, *i.e.*, at low  $\Delta K$  levels. The most ubiquitous form of wedging arises from fracture surface asperities (roughness-induced closure), but closure can also arise from the formation of oxide layers or other corrosion deposits created inside the crack (oxide-induced crack closure) (Ritchie, 1988). These latter closure mechanisms are not considered in this work which is solely focused on plasticity-driven fatigue crack growth.

#### Table 1

Model material properties and the values assigned to them in this study.

| Properties  | Symbol  | Value  |
|---|---|--|
| Young's modulus   | Е   | 200 GPa  |
| Initial yield stress  | $\sigma_0$  | 430 MPa  |
| Poisson's ratio   | ν   | 0.3  |
| Fatigue ductility coefficient in Coffin-Manson model                              | $\varepsilon'_{f}$  | 0.38 (Pereira et al., 2016)  |
| Fatigue ductility exponent in Coffin-Manson model                                 | c   | -0.56 (Pereira et al., 2016)   |
| Material constants for the evolution of back stress                               | $\begin{cases} C_1 \\ C_2 \\ C_3 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{cases}$ | $3 \times 10^{5}$ MPa (Chaboche, 1991)<br>$8 \times 10^{4}$ MPa (Chaboche, 1991)<br>$1.6 \times 10^{3}$ MPa (Chaboche, 1991)<br>$10^{4}$ (Chaboche, 1991)<br>800 (Chaboche, 1991)<br>1 |
| Material constants for the evolution of isotropic hardening/softening parameter R | {h<br>Q   | 8 (Chaboche, 1991)<br>70MPa  |



**Fig. 4.** Normalized accumulated plastic work density at failure,  $\bar{W}_{failure}^p/\sigma_0\varepsilon_0$ , in a cyclic uniaxial strain-controlled test as a function of the plastic strain amplitude,  $\Delta \varepsilon^p/2$ . The dashed line shows the asymptotic value. The yield stress is  $\sigma_0 = 430$  MPa and the yield strain  $\varepsilon_0 = \sigma_0/E = 2.15 \times 10^{-3}$ .

where  $\varepsilon'_f$  and *c* are the fatigue ductility coefficient and the fatigue ductility exponent, respectively. With a knowledge of these constants, for a given plastic strain amplitude, the number of cycles to failure can be simply calculated from Eq. (14). On the other hand, the amount of plastic work density accumulated per cycle  $\bar{W}^p_{cycle}$  can be obtained through:

$$\bar{W}^{p}_{cycle} = \oint_{cycle} \sigma_{ij} d\varepsilon^{p}_{ij}.$$
(15)

Under the assumption that the  $\bar{W}_{cycle}^{p}$  is constant for each cycle up to failure, the total plastic work density accumulation till failure under an applied plastic strain amplitude  $\Delta \varepsilon^{p}/2$  is given by:

$$\bar{W}_{failure}^{p}(\Delta\varepsilon^{p}/2) = \bar{W}_{cycle}^{p}(\Delta\varepsilon^{p}/2) \times N_{f}(\Delta\varepsilon^{p}/2).$$
(16)

For our model material, we carried out uniaxial strain-controlled cyclic simulations with plastic strain amplitudes varying from 0.1% to 20% to obtain the plastic work density  $\bar{W}^p_{cycle}$ . With the use of Eq. (16), the results for  $\bar{W}^p_{failure}$  are plotted in Fig. 4 which shows that as the plastic strain amplitude increases, the normalized total accumulated plastic work density for failure  $\bar{W}^p_{failure}/\sigma_0\varepsilon_0$  decreases to an asymptotic value  $D^*_0/\sigma_0\varepsilon_0 = 1300$ , for plastic strain amplitudes greater than about 15%. It is noted that the plastic strain amplitude at a crack tip during cyclic loading is much greater than 15%. Hence,  $D^*_0$  can be considered as an estimate of the critical value for the plastic dissipation to release a node. In our simulations with the C(T) specimen, we chose  $D_0 = 4D^*_0$  for the critical plastic dissipation parameter. As we elaborate in Section 4, parametric studies show that in order for a "reasonable" steady-state condition to be reached, the critical plastic dissipation  $D_0$  should be



**Fig. 5.** Schematic of the nodal release scheme relative to the loading cycle for the first 5 cycles in the C(T) specimen. The parameter  $K_i$  is the applied stress-intensity factor and N the cycle number.

about 3 to 5 times the plastic dissipation  $D_0^*$  obtained from the uniaxial strain-controlled simulation and the Coffin–Manson model.<sup>5</sup> We note that a "reasonable" approach to steady-state is one in which nodal release takes place such that the plastic accumulation at the crack tip is neither too excessive (too large a  $D_0$ ), which may lead to the shakedown situation that we discussed for the stationary crack, or too small in which case the resulting steady-state depends on the element size and the number of times nodes are released within a cycle of loading that is very large (too small a  $D_0$ ).

# 3.2. Numerical results

We present numerical results for the C(T) specimen by cycling the applied force *F* such that  $\Delta K = 20 \text{ MPa}\sqrt{\text{m}}$  with a load ratio (ratio of minimum to maximum loads) of zero, *i.e.*, for zero-to-tension cycling. Monotonic loading to the maximum stress intensity factor  $K_{\text{max}} = 20 \text{ MPa}\sqrt{\text{m}}$  generates a maximum plastic zone of size  $r_p = (1/3\pi)(K_{\text{max}}/\sigma_0)^2 = 0.23 \text{ mm}$  with a corresponding crack-tip opening displacement (CTOD)  $\delta_{CTOD} = 0.55 (K_{\text{max}}^2(1 - \nu^2)/\sigma_0 E) = 2.33 \mu\text{m}$ . The finite element mesh shown in Fig. 3, with element size  $0.12 \mu\text{m}$  for the near crack-tip elements, clearly resolves the CTOD at peak loading with about 20 element sides per  $\delta_{CTOD}$ . As mentioned before, in order to minimize numerical error due to severe mesh distortion, re-meshing was performed when the accumulated plastic strain from the last re-meshing reaches the critical value of 0.4. With this critical value, re-meshing was performed usually a couple of times within each cycle, both during the cycles with nodal release and during the last cycle without nodal release. Below we provide results for the establishment of the "steady-stress" condition which enables our determination of the fatigue crack growth rates.

#### 3.2.1. Steady-state crack growth

The steady-state crack growth condition in the C(T) specimen under cyclic loading was obtained using the nodal release technique. As we noted, whenever the accumulated plastic work density at the crack-tip element reaches the critical value  $D_0$ , the crack-tip node is released. Fig. 5 shows the initial five loading cycles in which the superposed dots represent the time within the cycles at which the crack-tip node is released. For the assumed critical plastic dissipation parameter and loading range, one node is released almost at every cycle until the steady-state condition is reached.

The crack profiles at various high and low peak loads toward steady-state are shown in Fig. 6. When the load is applied in the first cycle, the crack tip moves toward the load line, *i.e.*, to the left in the configuration of Fig. 6a, whereas upon unloading, the crack tip moves away from the load line, *i.e.*, to the right in Fig. 6b. The same trend is observed for other cycles, as shown for example, for the 10<sup>th</sup> cycle in Fig. 6c and d, the 20<sup>th</sup> cycle in Fig. 6e and f, and so on. Movement of the crack tip during loading and unloading is caused by the Poisson's effect; the real shifting of the crack tip between two cycles can be appreciated by comparing crack-tip locations, for example, at high or low peak cycles. Note that we observed the presence of crack closure at the tip even at the end of the first cycle. The cyclic loading is continued until a steady-state condition ahead of the crack tip is obtained; this is declared when the opening stress and plastic strain ahead of

<sup>&</sup>lt;sup>5</sup> It is worth noting that the critical plastic accumulation obtained from a macroscopic experiment in uniaxial tension is compared with the plastic accommodation locally at the crack tip. This is done under the premise that that the local crack-tip response is similar to macroscopic response. This may not be an unrealistic assumption due to the fact that the stress and strain fields in the Coffin–Manson test sample are uniform throughout and they essentially represent the plastic accommodation density. On the other hand, the number of cycles to failure determined through a Coffin–Manson test sample reflects the nature of the uniaxial stress field whose triaxiality is low in comparison to the levels at the crack tip. Although it is generally considered that such stress-state effects are relatively small in fatigue, nonetheless the number of cycles to failure that we used in Eq. (16) may need to be reconsidered. Perhaps the fact that we need to use in our simulations a critical plastic dissipation larger than the one provided by the Coffin–Manson model reflects the fact that the Coffin–Manson model underestimates the number of cycles to failure.



**Fig. 6.** Crack-tip profiles in the C(T) specimen for cycles N = 1, 10, 20, 40, and 80: (a), (c), (e), (g), and (i) at high peak when  $K_{max} = 20$  MPa $\sqrt{m}$ , and (b), (d), (f), (h), and (j) at low peak when  $K_{min} = 0$ . The arrows indicate the crack-tip location for each case and the dashed lines the location of the crack tip before loading.



**Fig. 7.** Comparison of (a) normalized opening stress  $\sigma_{yy}/\sigma_0$  and (b) effective plastic strain  $\varepsilon^p$  ahead of the crack tip at peak loading for cycles N=1, 40, 80, and 100 under cycling loading and  $K_{\min}=0$  to  $K_{\max}=20$  MPa $\sqrt{m}$ . The parameters r denotes the distance from the crack tip and b is the crack-tip opening displacement at the high peak of the corresponding cycle. The parameter b for cycles N=1, 40, 80, and 100 is 1.5, 0.8, 0.7, and 0.7 µm, respectively. The yield stress is  $\sigma_0 = 430$  MPa.

the crack tip at peak loading remains the same for several cycles while the crack advances. Fig. 7 shows the normalized opening stress  $\sigma_{yy}/\sigma_0$  and effective plastic strain  $\varepsilon^p$  as a function of the normalized distance r/b from the crack tip along the axis of symmetry for various load cycles. The parameter *b* denotes the CTOD defined by the 90° intersection method at the high peak loading for each cycle, which varies till the fields reach the steady-state. The CTODs at the high peak loadings for cycles N=1, 40, 80, and 100 with  $K_{\min}=0$  and  $K_{\max}=20$  MPa $\sqrt{m}$  are b=1.5, 0.8, 0.7, and 0.7 µm, respectively. As shown in Fig. 7, the normalized opening stress and plastic strain ahead of the crack tip for cycles N=80 and 100 are the same with b=0.7 µm for both cases. Therefore, we would conclude that the steady-state condition is reached at cycle 80.



**Fig. 8.** Plot of (a) the steady-state normalized opening stress  $\sigma_{yy}/\sigma_0$  and (b) the steady-state effective plastic strain  $\varepsilon^p$  at peak loading versus the normalized distance from the crack tip r/b under cyclic loading with  $K_{\text{max}} = 20 \text{ MPa}\sqrt{m}$  and  $K_{\text{min}} = 0$ . The parameter  $\sigma_0 = 430 \text{ MPa}$  is the initial yield stress and b = 0.7 µm is the crack-tip opening displacement at the peak loading at the steady-state condition.

The steady-state stress and plastic strain in the C(T) specimen subjected to cyclic loading with  $\Delta K = 20$  MPa $\sqrt{m}$  at a load ratio of zero is replotted in Fig. 8. As observed in Fig. 8a, the maximum opening stress ahead of the crack tip occurs at a distance about r = 1.2b from the tip. The insets in Fig. 8a and b show that the profiles of the opening stress and plastic strain have sections over which they vary as  $\ln(1/r)$  with distance r from the crack tip. It is interesting that such a plastic strain distribution is comparable to that obtained by Rice et al. (1980), Drugan et al. (1982), and Dean and Hutchinson (1980) for the non-stationary crack in an elastic perfectly plastic solid.<sup>6</sup> For the plastic strain, this relationship holds for b < r < 20b where 20b is approximately the plastic-zone size ahead of the crack tip. However, the  $\ln(1/r)$  variation of the opening stress ahead of the crack tip in the current solution for a fatigue crack is dominant over a shorter region. The deviation can be attributed to the fact that stress should match the elastic far-field when the plastic strain is very small.

# 3.2.2. Crack propagation rate at steady-state

Having obtained the steady-state condition for fatigue crack growth, we determine the crack growth rate by applying one additional load cycle while the nodal release is turned off. We note that similar to the case before reaching the steadystate, crack closure was observed at the crack tip at the end of this additional load cycle. We calculate the corresponding crack growth by comparing the locations of the crack tip at the low peaks, *i.e.*, at the beginning and the end of the cycle. Thus, for the growth rate for the cycle with  $\Delta K = 20$  MPa $\sqrt{m}$  at a load ratio of zero, we calculate  $da/dN = 0.05 \,\mu\text{m/cycle}$ . We use the same approach of the determination of the steady-state first to calculate the crack growth rate for various stress-intensity ranges at zero load ratio. Fig. 9 shows the da/dN results plotted as a function of  $\Delta K$ . As expected, the data exhibit Paris law behavior in the form of  $da/dN = C(\Delta K)^{\text{m}}$  where the scaling constants are determined to be m = 3.13and  $C = 4.17 \times 10^{-9} \,\text{mm/cycle/MPa}^{\text{m}}$ . A Paris exponent of  $m \sim 3$  is typical of most ductile metallic materials (Ritchie, 1999).

# 4. Discussion

We have presented a finite element methodology to calculate the plasticity-driven fatigue crack propagation rate under cyclic loading. Without introducing any specific failure criterion for crack propagation, our approach involves obtaining the steady-state condition for crack propagation under a given cyclic load through a nodal release technique. Once the steady-state condition is obtained the crack growth rate, da/dN, is calculated through an additionally imposed load cycle during which nodal release is suppressed.

<sup>&</sup>lt;sup>6</sup> Reviewing the work on the nature of the singularity in the near crack-tip solution, the works of Rice and Hutchison stand out. As discussed by Rice (1968, 1979), the dominant strain singularity for a growing crack in an elastic perfectly plastic solid is  $\ln(1/r)$ , unlike the approximately 1/r singularity for a stationary crack in a nonlinear elastic solid. Rice and his coworkers (Rice et al., 1980; Drugan et al., 1982) presented an analytical solution near the tip of an advancing crack in an elastic perfectly plastic material under plane-strain mode I and small-scale yielding conditions. Their analysis was for general quasi-static crack growth and not specific to the steady-state. The asymptotic solutions provided by Rice et al. (1980) and Drugan et al. (1982) showed a  $\ln(1/r)$  singularity for the plastic strain field. Dean and Hutchinson (1980), through finite element simulations of steady-state crack propagation in an elastic perfectly plastic material under small-scale yielding conditions for plane-strain mode I or mode III loading, also found the crack-tip dominant singularity for the strain fields to be  $\ln(1/r)$ . Our present simulations of crack propagation under cyclic loading also reveal steady-state strain solutions that exhibit an  $\ln(1/r)$  character, similar to the case of a non-stationary crack under monotonic loading.



**Fig. 9.** Plot of crack growth rate, da/dN, in a C(T) specimen as a function of the stress-intensity range,  $\Delta K$ , for cyclic loading with zero minimum stress-intensity factor,  $K_{\min} = 0$ , and  $10 \le K_{\max} \le 20$  MPa $\sqrt{m}$  (i.e., at a zero load ratio). The results display Paris law behavior with an exponent m = 3.13, which is typical of most ductile metallic materials (Ritchie, 1999).

To determine the steady-state condition, the crack-tip nodes are released when a measure of the accumulated plastic work at the crack tip reaches a critical value, namely when  $W^p = W^p_{cr}$ , where  $W^p$  so far in the calculations has been identified with the plastic dissipation density at the crack-tip element, and  $W^p_{cr}$  is the corresponding critical dissipation that was identified with the parameter  $D_0$  - a case that we term the *crack-tip element criterion*. In this section, we present parametric studies revealing i) the independence of the predicted crack growth rates da/dN from the way  $W^p$  is calculated, ii) the degree to which da/dN is sensitive to the critical magnitude of the plastic dissipation,  $W^p_{cr}$ , and iii) the independence of da/dN from the element size.

#### 4.1. Steady-state independence from the way the plastic dissipation W<sup>p</sup> is calculated

To explore the independence of the obtained steady-state from the way the plastic dissipation  $W^p$  is calculated with our nodal release criterion, we carried out calculations by considering  $W^p$  to be the dissipation: (a) along the entire row of elements along the symmetry line ahead of the crack tip – the *row of elements criterion*, and (b) from the entire plastic zone ahead of the crack tip – the *plastic-zone criterion*. As before, the calculations were carried out under cyclic loading with  $K_{\min} = 0$  and  $K_{\max} = 20$  MPa $\sqrt{m}$ . Since a larger number of elements is now involved in the calculation process, it was required that we adjust the critical plastic dissipation  $W_{Cr}^p$  accordingly for both cases (a) and (b). For the case of plastic dissipation along the row of elements, case (a), we were able to obtain an identical result to that achieved using the cracktip element criterion in Fig. 9 by choosing  $W_{Cr}^p = 5D_0A$  as the critical plastic dissipation parameter, where  $D_0$  is the critical value corresponding to the crack-tip element criterion and A is the crack-tip element area.

Fig. 10 shows the steady-state condition ahead of the crack tip at peak load,  $K_{\text{max}} = 20 \text{ MPa}\sqrt{\text{m}}$ , using the nodal release based on accumulated plastic work in the row of elements ahead of the crack tip with  $5D_0A$  as the critical parameter. The steady-state results corresponding to nodal release criterion of the accumulated plastic work at the crack-tip element reaching  $D_0A$  are also superposed on the figure. The similarity between the results from the crack-tip element criterion and the row of elements criterion confirms that, as discussed above, the criterion for nodal release is only a representative way of accounting for the plastic process ahead of the crack tip. We arrived at same conclusion when we used the *plastic-zone criterion* in the simulations.

# 4.2. Sensitivity of the steady-state on the critical value of the plastic dissipation $W_{cr}^p$

In order to obtain the steady-state condition, we considered a criterion for nodal release based on the accumulated plastic work density at the crack-tip element reaching a critical value. We used the Coffin–Manson model to obtain an estimate for the critical value  $D_0^* = 1300 \times \sigma_0 \varepsilon_0$  for the failure of material under uniaxial tension and presented numerical results for the steady-state and fatigue crack growth rate (Fig. 9) with the critical value for the plastic dissipation density  $D_0 = 4D_0^*$  for nodal release.

We performed a parametric study on the critical value  $D_0$  to evaluate the sensitivity of the predicted plasticity-driven crack propagation rate. The calculated crack growth rates are shown in Fig. 11 for various normalized critical plastic dissi-



**Fig. 10.** Comparison of the steady-state conditions under cyclic loading with  $K_{\text{max}} = 20$  MPa $\sqrt{m}$  and  $K_{\min} = 0$  obtained based on accumulated plastic work at the crack-tip element (dots) or over the row of elements ahead of the crack tip (solid lines): (a) Normalized steady-state opening stress  $\sigma_{yy}/\sigma_0$  and (b) steady-state effective plastic strain  $e^p$  along the symmetry line ahead of the crack tip at the peak loading plotted against the normalized distance r/b from the current crack tip. The yield stress is  $\sigma_0 = 430$  MPa and the crack-tip opening displacement at peak loading is  $b = 0.7 \,\mu\text{m}$ .



**Fig. 11.** Dependence of the fatigue crack propagation rate on the normalized critical value  $D_0/D_0^*$  for nodal release under cyclic loading with  $K_{\text{max}} = 20 \text{ MPa}_{\sqrt{m}}$  and  $K_{\min} = 0$ . The normalizing parameter  $D_0^* = 1300\sigma_0\varepsilon_0$  is the plastic dissipation density in a uniaxial strain-controlled test obtained from the Coffin-Manson model for plastic strain amplitudes greater than 15%,  $\sigma_0 = 430 \text{ MPa}$  is the yield stress and  $\varepsilon_0 = \sigma_0/E = 2.15 \times 10^{-3}$  is the yield strain.

pation values for nodal release. The figure shows that a smaller critical value results in a greater crack growth rate. It is important to realize that when the critical value is small, *e.g.*, when  $D_0 < 3D_0^*$ , the nodal release criterion is frequently met. For example, about 4 nodes were released in a load cycle when  $D_0 = D_0^*$ . Consequently, a small critical value does not allow enough development of plasticity ahead of the crack tip and the resulting steady-state condition is dominated by the numerical details of the nodal release technique. On the other hand, if the chosen critical value is too large, *e.g.*,  $D_0 > 5D_0^*$ , the nodal release criterion is rarely met within a single loading cycle and excessive amount of plasticity develops ahead of the crack tip. Our parametric studies on an optimum critical value for nodal release indicate that with  $3D_0^* < D_0 < 5D_0^*$  nodes are released at a rate one per cycle.

Lastly, we investigated the sensitivity of the predicted crack propagation rate to the critical plastic dissipation magnitude,  $D_0$ , by calculating the steady-state rate also for the stress-intensity range  $\Delta K = 12.5$  MPa $\sqrt{m}$  at zero load ratio. The results for  $D_0$  equal to  $D_0^*$ ,  $2D_0^*$ ,  $4D_0^*$ , and  $6D_0^*$  under  $\Delta K = 12.5$  and 20 MPa $\sqrt{m}$  at zero load ratio are shown in Fig. 12. One may conclude that the variation of the crack growth rate with increasing  $D_0$  is the same for both  $\Delta K = 12.5$  and 20 MPa $\sqrt{m}$ . We also observe that the crack growth rate values tend to converge as  $D_0$  gets larger than  $4D_0^*$ . Given that the calculations of the crack propagation rate throughout this work were carried out based on the assumption that  $D_0 = 4D_0^*$ , we may deem the model predictions as only weakly dependent on the choice of  $D_0$  when  $3D_0^* < D_0 < 5D_0^*$ .



**Fig. 12.** Crack growth rate sensitivity to the critical plastic dissipation  $D_0$  under cyclic loading with  $\Delta K = 12.5$  and 20 MPa $\sqrt{m}$  and zero minimum stress-intensity factor.



**Fig. 13.** Effect of mesh size on the steady-state condition: (a) normalized steady-state opening stress  $\sigma_{yy}/\sigma_0$ , and (b) steady-state effective plastic strain  $\varepsilon^p$  along the axis of symmetry ahead of the crack tip at peak load under cyclic loading with  $K_{\text{max}} = 20$  MPa $\sqrt{m}$  and  $K_{\text{min}} = 0$ . The results are plotted against the normalized distance from the current crack tip r/b, where  $b = 0.7 \,\mu\text{m}$  is the crack-tip opening displacement. The parameter  $\delta_{CTOD}$  is the crack-tip opening displacement under monotonic loading, *i.e.*,  $0.55 \, (K_{\text{max}}^2 (1 - \nu^2)/\sigma_0 E)$  and  $\sigma_0 = 430$  MPa is the yield stress.

# 4.3. Mesh sensitivity

To investigate the effect of mesh density on the steady-state condition, we performed a mesh sensitivity study by comparing the results obtained with element sizes of 0.12, 0.18, 0.24, and 0.3 µm close to the crack tip under cyclic loading with  $\Delta K = 20$  MPa $\sqrt{m}$  at zero load ratio. It is noted that with element sizes of 0.12, 0.18, 0.24, and 0.3 µm, the CTOD under monotonic loading up to the maximum load (*i.e.*,  $\delta_{CTOD} = 0.55$  ( $K_{max}^2(1 - v^2)/\sigma_0 E$ ) = 2.33 µm) is resolved with about 20, 13, 10, and 8 elements, respectively. The opening stress and plastic strain ahead of the crack tip are shown in Fig. 13 for various element sizes at the peak load,  $K_{max} = 20$  MPa $\sqrt{m}$ , and after the steady-state condition was reached. The results display convergence to the steady-state as the element size decreases.

Using the established steady-state condition, we determined the crack growth rates corresponding to  $K_{\text{max}} = 20 \text{ MPa}\sqrt{\text{m}}$  for each element size by applying one additional load cycle while the nodal release was turned off. The approach was repeated for additional stress intensity ranges with  $K_{\text{max}} = 12.5 \text{ MPa}\sqrt{\text{m}}$  at zero load ratio. Fig. 14 shows the corresponding crack growth rate results for various element sizes. The results indicate that the obtained crack growth rate also converges as the element size decreases.



**Fig. 14.** Calculated fatigue crack growth rate, da/dN, for a compact-tension specimen of a model steel under cyclic loading with  $\Delta K$  = 12.5 and 20 MPa $\sqrt{m}$  with various element sizes. The parameter  $\delta_{CTOD}$  (=0.55 ( $K_{max}^2(1 - \nu^2)/\sigma_0 E$ ) = 2.3 µm) is the crack-tip opening displacement under monotonic loading based on elastic-plastic fracture mechanics.

## 5. Summary and conclusions

We are proposing a numerical methodology for calculating the plasticity-driven fatigue crack growth rate in ductile materials. The features of our approach are summarized as follows:

- (i) The steady-state crack growth rate *da/dN* is calculated by considering the change in the location of the crack tip during a single load cycle. Significantly, this calculation of the crack growth rate does not involve any failure criterion for crack propagation. The magnitude of the crack growth rate reflects the plastic state of the material ahead of the crack tip and the associated plastic dissipation. At a given level of load during the load cycle, *e.g.*, at peak load, the steady-state is characterized by stress and strain fields ahead of the propagating crack tip that do not change with crack size.
- (ii) In order to determine the steady-state rate of cyclic crack advance per cycle, we employ the nodal release technique. A node is released when a measure of the plastic dissipation ahead of the crack tip reaches a critical value.
- (iii) For the plastic constitutive response of the material to cyclic loading, we adopted the Chaboche model because it can reproduce most of the essential characteristics of the material deformation under cyclic loading.

The method was implemented in calculating the fatigue crack growth rate in a compact-tension C(T) specimen for a model material. Within a load cycle under a given stress-intensity range, with the load ratio equal to zero, the large strain elastoplastic boundary value problem was solved incrementally by properly treating crack closure. A method of re-meshing the element layout near the crack tip was used to reduce numerical error when element distortion was severe. The simulation results may be summarized as follows:

- (i) The nodal release for crack advance toward steady-state took place when the plastic dissipation in the neighborhood of the crack tip reached a critical value. This criterion can be expressed in terms of the accumulated plastic work at the crack-tip element, or the row of elements along the axis of symmetry ahead of the crack, or the entire plastic zone. By adjusting the critical value for the plastic dissipation in the nodal release criterion, one can get the same steady-state fields irrespective of how the accumulated plastic work is calculated. In other words, the criterion for nodal release reflects a measure of the crack-tip plastic dissipation. In addition, the attainment of the steady-state is only weakly dependent on the choice of the magnitude of the critical plastic dissipation, when it is chosen to be about four times the dissipation as calculated from uniaxial cyclic tension.
- (ii) Parametric studies showed that the steady-state fields, reached through nodal release, and the calculated fatigue crack growth rate, found through one cycle of load after the steady-state is reached, are not sensitive to the element size near the crack tip, provided the mesh resolves the crack-tip opening displacement.
- (iii) The profiles of the steady-state strain fields ahead of the crack tip in a C(T) specimen involve a region in which they vary as ln(1/r) with distance r from the crack tip, behavior similar to the case of a non-stationary crack propagating in an elastic perfectly plastic material under a monotonic load.
- (iv) The calculated crack growth rate as a function of the stress-intensity range exhibited Paris law behavior with exponent of 3.13, which is typical value for a ductile metallic material.

The proposed mechanistic approach for the prediction of the fatigue crack growth rate is based on parameters that can be measured through independent experiments: For the description of the plastic deformation of the material under cyclic loading, a proper constitutive law is needed, *e.g.*, the Chaboche model. Calibration of the Chaboche model requires a sequence of uniaxial strain-controlled cyclic plasticity experiments as well as ratcheting tests. As for the critical value of the plastic dissipation for the nodal release, the plastic work from tests such as uniaxial cyclic plasticity can be used. Again, we note that the approach does not require a failure criterion for the advancement of crack growth; the steady-state crack growth rate is calculated during a loading cycle that involves solely the recognized process of plasticity-driven fatigue crack advance, that of alternating crack-tip blunting and re-sharpening.

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# Appendix A. Behavior of static crack under cyclic loading

Several researchers, e.g. (Gu and Ritchie, 1999; Tvergaard and Hutchinson, 2002; Tvergaard, 2004;Ellyin and Wu, 1992; Wu and Ellyin, 1996; Jiang and Feng, 2004), have studied the stress and strain fields at a stationary crack under cyclic loading for the purpose of describing crack growth behavior. To assess this approach, in this Appendix, we present the stress and strain fields in the neighborhood of a stationary crack in a C(T) specimen subjected to a cyclic loading. The geometry, the material constitutive law, and the finite element procedures are the same as those described in Section 3 on Numerical Implementation. The only difference in these calculations is that nodal release is switched off so that the crack remains stationary. The schematic of the C(T) specimen and finite element mesh are shown in Figs. 2 and 3, respectively. The crack was initially sharp with a size  $a_0 = 7.1$  mm. The specimen was subjected to cyclic loading between zero stress intensity,  $K_{\min} = 0$ , and maximum stress intensity  $K_{max} = 20$  MPa $\sqrt{m}$ .

Fig. A1 shows the normalized opening stress  $\sigma_{yy}/\sigma_0$  and effective plastic strain  $\varepsilon^p$  against normalized distance r/b from the crack tip at peak loading for various cycles. As observed from Fig. A1a, the crack-tip stress attains its maximum at the peak load of the first cycle and decreases in subsequent cycles. This behavior is dictated by how the back stress changes as plastic strain accumulates. At the crack tip, the back stress reaches a maximum at the peak load of the first cycle and thereafter it decreases over the following cycles. Consequently, by virtue of the yield condition, Eq. (6), the opening stress at the crack tip is the highest at the first cycle.

In contrast to the opening stress, the plastic strain ahead of the crack tip increases with the number of cycles, and as shown in Fig. A1b, it can reach abnormally high values, for instance, after 60 cycles. In other words, continued application of cyclic loading to a stationary crack leads to continued and unrealistic plastic straining for a structural material. This result indicates that the use of stationary crack calculations under cyclic loading to find the steady-state condition and thereby predicting fatigue crack growth rate should be avoided.



**Fig. A1.** Profiles of (a) normalized opening stress  $\sigma_{yy}/\sigma_0$  and (b) effective plastic strain  $\varepsilon^p$  ahead of a stationary crack tip at peak load for cycles N=1, 10, 20, 40, and 60 under  $\Delta K=20$  MPa $_{\sqrt{m}}$  and zero load ratio. The parameters r and b are the distance from the current crack tip and the crack-tip opening displacement at each cycle, respectively. The yield stress is  $\sigma_0 = 430$  MPa.

#### Appendix B. Re-meshing procedure

In this appendix, we present the numerical procedure implemented in our simulation for the crack-tip re-meshing when straining of the crack-tip elements becomes severe. As discussed in Section 3 on numerical simulation, we performed a re-meshing procedure to minimize numerical error due to severe mesh distortion resulting from large accumulated plastic strains near a fatigue crack. The re-meshing procedure was carried out when the accumulated plastic strain from the last re-meshing reached a critical value (Tvergaard, 2004), which was taken to be 0.4 in our simulations.

When the re-meshing condition was met, a new mesh was generated. For each new nodal point, the local coordinate with respect to the previous mesh was determined by finding the previous mesh element that contains the new nodal point. We then calculated the stress and strain fields as well as the state variables in the new nodal points from projection of those values from the previous mesh. Lastly, the quantities at the new integration points of each element were projected using the values at the new nodal points of each element.

## Appendix C. The Chaboche constitutive model with three kinematic hardening rules: uniaxial cyclic strain response

Experimental observations show that the stable stress-strain hysteresis loop for uniaxial cyclic loading under a straincontrolled condition can be divided into three different segments (Bari and Hassan, 2000; Hassan and Kyriakides, 1992):

- Segment 1 represents the part of the stable hysteresis curve with high modulus at very small strains and the onset of yielding;
- Segment 2 represents the transient nonlinear portion of the hysteresis curve at moderate plastic strains;
- Segment 3 represents the subsequent part of the hysteresis curve at high plastic strains.

Accordingly, a suitable nonlinear kinematic hardening model for cyclic loading should exhibit a hysteresis loop consisting of these three segments.

Although the model of Armstrong and Frederick (Armstrong and Frederick, 1966) is the seminal nonlinear kinematic hardening model in which  $\dot{\alpha} = (2/3)C \mathbf{D}^p - \gamma \alpha \dot{\varepsilon}^p$ , it cannot accurately represent the hysteresis loop segments as observed in fatigue experiments. To improve the prediction of the stress-strain hysteresis loop in a strain-controlled test, Chaboche and co-workers (Chaboche et al., 1979; Chaboche, 1986) proposed a nonlinear kinematic hardening model which is a superposition of three Armstrong and Frederick hardening rules, *i.e.*,  $\alpha = \alpha_1 + \alpha_2 + \alpha_3$  with evolution rules given by Eq. (7). The three hardening rules were chosen such that each one underlies a corresponding segment of the hysteresis loop. The first rule is designed to give initial hardening with a large modulus at the very small plastic strains followed by quick stabilization. The second rule describes the nonlinear behavior moderate plastic strains and which stabilizes at, or prior to, the maximum (minimum) plastic strain in the loading (unloading) part of the curve. The third hardening rule represents the near linear segment of the hysteresis loop along which the modulus is constant. Therefore, this rule should have  $\gamma_3$  much smaller than  $\gamma_1$  and  $\gamma_2$ .

To demonstrate the features associated with the three kinematic hardening rules, we simulate the uniaxial straincontrolled response to symmetric cyclic loading *i.e.*,  $\varepsilon_{max}^p = -\varepsilon_{min}^p$ . For such a loading, Eq. (7) reduces to:

$$\dot{\alpha}_i = C_i \dot{\varepsilon}^p - \lambda \gamma_i \alpha_i \dot{\varepsilon}^p, \quad i = 1, 2, 3 \text{(no summation over } i\text{)}. \tag{C1}$$

where  $\dot{\alpha}_i$  represents 3/2 of the normal component of the back-stress tensor in the loading direction and  $\lambda$  is the loading/unloading parameter such that  $\lambda = 1$  for loading and  $\lambda = -1$  for unloading. (Note that  $\overset{\nabla}{\alpha}_i = \dot{\alpha}_i$  under uniaxial loading since the spin tensor is zero). One can solve Eq. (C1) for the back stress  $\alpha_i$  to get:

$$\alpha_{i} = \lambda \frac{C_{i}}{\gamma_{i}} + \left(\alpha_{0} - \lambda \frac{C_{i}}{\gamma_{i}}\right) \exp\left(-\lambda \gamma_{i} \left(\varepsilon^{p} - \varepsilon_{0}^{p}\right)\right), i = 1, 2, 3,$$
(C2)

where  $\alpha_0$  and  $\varepsilon_0^p$  are the values of  $\alpha_i$  and  $\varepsilon^p$  at the start of loading or unloading. Knowing that the first and second back stresses,  $\alpha_1$  and  $\alpha_2$ , when stabilized have the same tensile and compressive values within the strain range of the stable loop, one obtains  $\alpha_1$  and  $\alpha_2$  as:

$$\alpha_{i} = \lambda \frac{C_{i}}{\gamma_{i}} \left( 1 - 2 \exp\left(-\lambda \gamma_{i} \left(\varepsilon^{p} - \lambda \varepsilon_{\min}^{p}\right)\right) \right), \quad i = 1, 2 \text{(no summation over } i\text{)}.$$
(C3)

In addition, since  $\gamma_3$  is small, the third back stress,  $\alpha_3$ , is considered to be almost linear such that it passes through the origin. Hence, the third back stress can be written as:

$$\alpha_3 = \frac{\zeta_3}{\gamma_3} (1 - \exp\left(-\gamma_3 \varepsilon^p\right)),\tag{C4}$$

and since  $\gamma_3$  is small, Eq. (C4) can be approximated by:

$$\alpha_3 \approx C_3 \varepsilon^p. \tag{C5}$$

Fig. C1 shows the results for the stable hysteresis loop when  $\varepsilon_{\max}^p = -\varepsilon_{\min}^p = 1\%$  of the Chaboche model material whose material constants  $C_i$  and  $\gamma_i$  are given in Table 1. The first back stress,  $\alpha_1$ , starts with a large initial slope and then stabilizes



**Fig. C1.** The stable stress-strain hysteresis loop and associated back stress for the Chaboche model in uniaxial strain-controlled cyclic loading with plastic strain bounds  $\pm$  1%. The model's parameters are presented in Table 1.

to  $C_1/\gamma_1$  ( $-C_1/\gamma_1$ ) in the loading (unloading) part of the curve. Similarly, the second back stress stabilizes to  $C_2/\gamma_2$  ( $-C_2/\gamma_2$ ) in the loading (unloading) part of the curve and, as can be seen from Fig. C1, it simulates the nonlinear portion of the stable hysteresis curve. The third back stress starts from  $C_3 \varepsilon_{\min}^p$  ( $C_3 \varepsilon_{\max}^p$ ) in the loading (unloading) part and passes through the origin. It clearly represents the linear part of the stable hysteresis curve.

# Appendix D. Numerical integration of the Chaboche constitutive model

Following Govindarajan and Aravas (1994a,b), we seek the solution ( $\sigma_{n+1}$ ,  $\alpha_{n+1}$ ,  $R_{n+1}$ ,  $\varepsilon_{n+1}^p = \varepsilon_n^p + \Delta \varepsilon^p$ ) at time  $t_{n+1} = t_n + \Delta t$  when the solution ( $\mathbf{F}_n$ ,  $\sigma_n$ ,  $\alpha_n$ ,  $R_n, \varepsilon_n^p$ ) at time  $t_n$  and the deformation gradient  $\mathbf{F}_{n+1}$  are known. The deformation gradient tensor at any time  $t \in [t_n, t_{n+1}]$  can be written as:

$$\mathbf{F}(t) = \Delta \mathbf{F}(t) \mathbf{F}_n = \mathbf{R}(t) \mathbf{U}(t) \mathbf{F}_n, \qquad t_n \le t \le t_{n+1}$$
(D1)

where  $\mathbf{R}(t)$  and  $\mathbf{U}(t)$  are the rotation and right stretch tensors respectively associated with  $\Delta \mathbf{F}(t)$ . The right stretch tensor is written as:

$$\mathbf{U}(t) = \sum_{i=1}^{3} \lambda_i(t) \, \mathbf{n}_i \otimes \mathbf{n}_i, \tag{D2}$$

where  $\lambda_i(t)$  and  $\mathbf{n}_i$  are the corresponding eigenvalues and eigenvectors of  $\mathbf{U}(t)$  for  $t_n < t < t_{n+1}$ , and  $\otimes$  denotes tensor product. The eigenvectors  $\mathbf{n}_i$  are assumed to be constant during the time increment,  $t_n < t < t_{n+1}$ . The deformation rate tensor  $\mathbf{D}(t)$  and the spin tensor  $\mathbf{W}(t)$  of  $\mathbf{F}(t)$  can then be calculated through:

$$\mathbf{D}(t) = [\mathbf{F}(t)\mathbf{F}^{-1}(t)]_{sym} = \mathbf{R}(t)\mathbf{E}(t)\mathbf{R}^{T}(t), \tag{D3}$$

$$\mathbf{W}(t) = [\dot{\mathbf{F}}(t)\mathbf{F}^{-1}(t)]_{asym} = \dot{\mathbf{R}}(t)\mathbf{R}^{\mathrm{T}}(t), \tag{D4}$$

where []<sub>sym</sub> and []<sub>asym</sub> denote respectively symmetric and antisymmetric parts and  $\mathbf{E}(t)$  is the logarithmic strain relative to the configuration at time  $t_n$  given by:

$$\mathbf{E}(t) = \ln \mathbf{U}(t) = \sum_{i=1}^{3} \ln(\lambda_i) \, \mathbf{n}_i \otimes \mathbf{n}_i.$$
(D5)

The Jaumann rate of the Cauchy stress tensor and the *i*<sup>th</sup> back stress tensor are defined as:

$$\check{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma} \boldsymbol{W} - \boldsymbol{W} \boldsymbol{\sigma},\tag{D6}$$

and

$$\stackrel{\vee}{\boldsymbol{\alpha}}_{i}=\dot{\boldsymbol{\alpha}}+\boldsymbol{\alpha}_{i}\boldsymbol{W}-\boldsymbol{W}\boldsymbol{\alpha}_{i},\tag{D7}$$

respectively. Defining the co-rotational stress tensors  $\hat{\sigma}$  and  $\hat{\alpha}_i$  through:

$$\widehat{\boldsymbol{\sigma}} = \mathbf{R}^T \boldsymbol{\sigma} \mathbf{R},$$

$$\widehat{\boldsymbol{\alpha}}_i = \mathbf{R}^T \boldsymbol{\alpha}_i \mathbf{R},$$
(D8)

one can show that:

$$\hat{\boldsymbol{\sigma}} = \mathbf{R}^T \stackrel{\nabla}{\boldsymbol{\sigma}} \mathbf{R},$$

$$\hat{\boldsymbol{\alpha}}_i = \mathbf{R}^T \stackrel{\nabla}{\boldsymbol{\alpha}}_i \mathbf{R}.$$
(D9)

Using Eqs. (D3) and (D9), we can write Eqs. (3), (4), (6)-(9), and Eq. (11) as

$$\dot{\mathbf{E}} = \dot{\mathbf{E}}^e + \dot{\mathbf{E}}^p$$
.

$$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^p,\tag{D10}$$

$$\widehat{\boldsymbol{\sigma}} = \mathbf{C}^e : \dot{\mathbf{E}}^e, \tag{D11}$$

$$f(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\alpha}}, R) = \widehat{\Sigma}_e - \sigma_0 - R = 0, \tag{D12}$$

$$\hat{\boldsymbol{\alpha}}_{i} = \frac{2}{3} C_{i} \dot{\mathbf{E}}^{p} - \gamma_{i} \widehat{\boldsymbol{\alpha}}_{i} \dot{\boldsymbol{\varepsilon}}^{p}, \quad i = 1, 2, 3, \tag{D13}$$

$$\dot{\varepsilon}^p = \frac{2}{3} \dot{\mathbf{E}}^p : \dot{\mathbf{E}}^p, \tag{D14}$$

$$\dot{\mathbf{E}}^p = \dot{\varepsilon}^p \mathbf{\hat{N}},\tag{D15}$$

$$\dot{R} = h(Q - R)\dot{\varepsilon}^p,\tag{D16}$$

where we defined  $\widehat{\mathbf{N}} = 3(\widehat{\boldsymbol{\sigma}}' - \widehat{\boldsymbol{\alpha}})/2\widehat{\Sigma}_e$  in which  $\widehat{\Sigma}_e = \sqrt{(3/2)(\widehat{\boldsymbol{\sigma}}' - \widehat{\boldsymbol{\alpha}}) : (\widehat{\boldsymbol{\sigma}}' - \widehat{\boldsymbol{\alpha}})}$ . Thus, the problem has been reduced to: given the solution  $\widehat{\boldsymbol{\sigma}}_n$ ,  $\widehat{\boldsymbol{\alpha}}_n$ ,  $R_n \varepsilon_n^p$  at time  $t_n$  and the logarithmic strain increment  $\Delta \mathbf{E}$  over the time interval  $[t_n, t_{n+1}]$ , we seek the solution  $\widehat{\boldsymbol{\sigma}}_{n+1}$ ,  $\widehat{\boldsymbol{\alpha}}_{n+1}$ ,  $R_{n+1}$ ,  $\varepsilon_{n+1}^p = \varepsilon_n^p + \Delta \varepsilon^p$  at time  $t_{n+1}$ . Eqs. (D10), (D11), (D13), and (D16) can be integrated over the given time increment  $\Delta t$  to give:

$$\widehat{\boldsymbol{\sigma}}_{n+1} = \widehat{\boldsymbol{\sigma}}^e - \mathbf{C}^e : \Delta \mathbf{E}^p, \tag{D17}$$

$$(\widehat{\boldsymbol{\alpha}}_i)_{n+1} = \frac{(\widehat{\boldsymbol{\alpha}}_i)_n + \frac{2}{3}C_i\Delta \mathbf{E}^p}{1 + \gamma_i\Delta\varepsilon^p}, \quad i = 1, 2, 3,$$
(D18)

$$R_{n+1} = \frac{R_n + hQ\Delta\varepsilon^p}{1 + \Delta\varepsilon^p},\tag{D19}$$

where  $\hat{\sigma}^e = \hat{\sigma}_n + \mathbf{C}^e$ :  $\Delta \mathbf{E}$  is the elastic predictor. Backward Euler integration of Eq. (D15) yields the plastic strain increment  $\Delta \mathbf{E}^p$  as:

$$\Delta \mathbf{E}^p = \Delta \varepsilon^p \mathbf{\hat{N}}_{n+1},\tag{D20}$$

where

$$\widehat{\mathbf{N}}_{n+1} = \frac{3}{2} \frac{\left(\widehat{\boldsymbol{\sigma}}^{e'} - \widehat{\boldsymbol{\theta}}\right)}{\widehat{\Sigma}_{\theta e}},\tag{D21}$$

 $\widehat{\boldsymbol{\theta}}$  is function of the effective plastic strain increment given by:

$$\widehat{\boldsymbol{\theta}}(\Delta \varepsilon^p) = \sum_{i=1}^{3} \frac{(\widehat{\boldsymbol{\alpha}}_i)_n}{1 + \gamma_i \Delta \varepsilon^p},\tag{D22}$$

and

$$\widehat{\Sigma}_{\theta e} = \sqrt{\frac{3}{2} \left( \widehat{\boldsymbol{\sigma}}^{e'} - \widehat{\boldsymbol{\theta}} \right) : \left( \widehat{\boldsymbol{\sigma}}^{e'} - \widehat{\boldsymbol{\theta}} \right)}.$$
(D23)

We introduce

$$\widehat{\Sigma}_{e}^{e} = \sqrt{\frac{3}{2}} \left( \widehat{\boldsymbol{\sigma}}^{e'} - \widehat{\boldsymbol{\alpha}}_{n} \right) : \left( \widehat{\boldsymbol{\sigma}}^{e'} - \widehat{\boldsymbol{\alpha}}_{n} \right), \tag{D24}$$

361

to determine whether the deformation over the current step involves additional plastic deformation. If  $\hat{\Sigma}_e^e - \sigma_0 - R_n \le 0$ , the deformation increment in the current step is elastic, *i.e.*,  $\Delta \varepsilon^p = 0$ . Hence, the stress tensor, back stress tensor, and isotropic parameter at the end of the step can be obtained as  $\hat{\sigma}_{n+1} = \hat{\sigma}^e$ ,  $\hat{\alpha}_{n+1} = \hat{\alpha}_n$ , and  $R_{n+1} = R_n$ , respectively. If  $\hat{\Sigma}_e^e - \sigma_0 - R_n > 0$ , plastic straining takes place during the time increment. The yield condition at time  $t_{n+1}$  can be written as:

$$f = (\hat{\Sigma}_e)_{n+1} - \sigma_0 - R_{n+1} = 0, \tag{D25}$$

where  $(\hat{\Sigma}_e)_{n+1}$  and  $R_{n+1}$  are nonlinear functions of the effective plastic strain increment  $\Delta \varepsilon^p$ . Eq. (D25) is solved numerically through Newton iteration to obtain  $\Delta \varepsilon^p$ . Using the effective plastic strain increment  $\Delta \varepsilon^p$ , one can obtain  $\Delta \mathbf{E}^p$  from Eq. (D20) and finally  $\hat{\sigma}_{n+1}$ ,  $(\hat{\alpha}_i)_{n+1}$  and  $R_{n+1}$  at time  $t_{n+1}$  can be determined from Eqs. (D17), (D18), and (D19), respectively. The consistent elastoplastic tangent moduli are defined as:

$$\mathbf{C}^{c} = \left(\frac{\partial \Delta \boldsymbol{\sigma}}{\partial \Delta \mathbf{E}}\right)_{n+1},\tag{D26}$$

in which  $\Delta \sigma$  is the stress increment over the time interval  $[t_n, t_{n+1}]$  and  $\Delta \mathbf{E}$  is the associated strain increment. The consistent moduli are obtained at the end of step at time  $t_{n+1}$ . Eqs. (D10) and (D11) may be integrated to give the co-rotational stress increment as:

$$\Delta \widehat{\boldsymbol{\sigma}} = \mathbf{C}^e : \Delta \mathbf{E} - \mathbf{C}^e : \Delta \mathbf{E}^p. \tag{D27}$$

Differentiating  $\Delta \hat{\sigma}$  with respect to  $\Delta E$  and using Eq. (B8), one can obtain the consistent moduli as:

$$\mathbf{C}^{c} = \mathbf{C}^{e} + \frac{2G^{2}\Delta\varepsilon^{p}}{\Sigma_{\theta e}}(2\mathbf{N}_{n+1}\mathbf{N}_{n+1} - 3\mathbf{K}) - \frac{4G^{2}}{3G + \mathbf{N}_{n+1} : \frac{\partial\boldsymbol{\alpha}_{n+1}}{\partial\Delta\varepsilon^{p}} + \frac{\partial\boldsymbol{R}_{n+1}}{\partial\Delta\varepsilon^{p}}}(\mathbf{N}_{n+1} - \frac{\Delta\varepsilon^{p}}{\Sigma_{\theta e}}\boldsymbol{\xi})\mathbf{N}_{n+1},$$
(D28)

where **K** is the deviatoric part of the forth-order identity tensor,  $\Sigma_{\theta e} = \hat{\Sigma}_{\theta e}$ , and

$$\boldsymbol{\xi} = \frac{3}{2} \frac{\partial \boldsymbol{\theta}}{\partial \Delta \varepsilon^p} - \left( \mathbf{N}_{n+1} : \frac{\partial \boldsymbol{\theta}}{\partial \Delta \varepsilon^p} \right) \mathbf{N}_{n+1}, \tag{D29}$$

in which  $\theta$  is given by:

$$\boldsymbol{\theta} = \sum_{i=1}^{3} \frac{(\boldsymbol{\alpha}_i)_n}{1 + \gamma_i \Delta \varepsilon^p}.$$
(D30)

#### Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.jmps.2018.07.026.

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